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1983 J. Phys. A: Math. Gen. 16 3533

(http://iopscience.iop.org/0305-4470/16/15/018)

J. Phys. A: Math. Gen. 16 (1983) 3533–3543. Printed in Great Britain

Fluctuations in the general relativistic theory of fluids and solids

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Received 12 April 1983

Abstract. A formalism to describe small random fluctuations in general relativistic continuous media is presented. It is based on the scheme of general relativistic irreversible thermodynamics and follows the lines of the non-relativistic theory of hydrodynamic fluctuations by Landau and Lifshitz. In a natural way Einstein's field equations and the equation of motion take a form which in principle allows us to calculate the mean behaviour and the correlation functions of any physical quantity of the theory. As an example fluctuations of a long thin bar under the influence of weak gravitational fields are treated.

1. Introduction

In the past decade the theoretical treatment of fluctuations in non-relativistic continuous media became of considerable interest (Fox and Uhlenbeck 1970, Keizer 1978, Ueyama 1980, Brenig and van den Broeck 1980, Kac and Logan 1979, Fox 1978, van Kampen 1976). Since it yields a refinement of the usual pure phenomenological laws of matter due to its particle structure, it may be regarded as some kind of link between the macroscopic and microscopic points of view. Whereas macroscopic laws describe a mean behaviour of a complicated many-particle system, fluctuation theory investigates small deviations from this mean behaviour, establishes probability laws and calculates correlation functions for the macroscopic variables.

An extension of these ideas to the theory of relativity would be of interest for several reasons. Firstly it is of general theoretical interest, secondly one could use it in actual problems of cosmology and astrophysics, and finally the random fluctuations in experimental devices for the detection of gravitational effects could be treated within a closed theoretical framework rather than *ad hoc* as is done in most current investigations (Braginsky and Manukin 1974).

On the other hand, it is well known that a satisfactory general relativistic Γ-space-statistics does not exist at present. Therefore at first glance it seems to be impossible to unify the concepts of general relativity with those of fluctuations of the phenomenological quantities in a closed theoretical framework. Since the fluctuations are due to the particle structure of matter it is unclear how to calculate quantities like correlation functions and mean square values.

Fortunately there exists the possibility of an approach to relativistic fluctuation problems which does not necessarily involve a relativistic many-particle theory. This approach is based on the ideas of the non-relativistic theory of hydrodynamic fluctuations by Landau and Lifshitz (1957, 1966). The basic variables of the Landau–Lifshitz theory are those of non-relativistic hydrodynamics. The theory enables us to calculate mean values and correlation functions of the phenomenological hydrodynamic quantities. Once derived, either by the help of probability theory (Fox and Uhlenbeck 1970) or from non-relativistic many-particle theory (Ueyama 1980), the fluctuation theory works on an entirely phenomenological level. Therefore it can be regarded as well as an axiomatic base of a purely phenomenological theory of fluctuations. Taking this point of view, it will be shown that a general relativistic counterpart of the Landau–Lifshitz theory of hydrodynamic fluctuations can be developed.

In the formulation of this fluctuation theory the scheme of general relativistic irreversible thermodynamics given by Neugebauer (1977, 1980) is extremely useful. Within this scheme the application of the strong equivalence principle of Einstein (Misner *et al* 1973) to the theory of Landau and Lifshitz is possible. Moreover it allows a unified description of all continuous media. While the original non-relativistic fluctuation theory (Landau and Lifshitz 1957) is confined to hydrodynamics, the present paper comprehends both fluids and solids. To our knowledge also in non-relativistic theory fluctuations in solids have never been treated in a corresponding framework. Principally it is possible to calculate correlation functions of all the matter quantities and the metric functions.

The 'fluctuational forces' which according to Landau and Lifshitz occur in the linear phenomenological relations for the heat flow vector and the viscosity tensor respectively become components of a four-dimensional tensorial quantity being the fluctuational part of the energy–momentum tensor with certain correlation properties.

The theory will be specialised to irreversible processes in relativistic fluids and solids. For the latter case an example of experimental interest, fluctuations in a somewhat idealised Weber bar in the field of a weak gravitational wave, is presented.

2. General theory

The essential points of the Landau–Lifshitz theory of small fluctuations are:

(i) adding of random terms to the linear phenomenological laws for the heat flow vector and the viscosity tensor, respectively;

(ii) determination of space–time correlation functions of these random terms.

By the help of (i) and (ii) one is able to calculate the correlation functions of all physical quantities of the theory, e.g. of density, pressure and temperature.

The origin of the fluctuations can be explained only in the framework of a microscopic many-particle theory (Keizer 1978, Ueyama 1980, Brenig and van den Broeck 1980).

A convenient starting point for a relativistically generalised method to incorporate small fluctuations into the phenomenological theory is the formulation of covariant irreversible thermodynamics given by Neugebauer (1977, 1980). Neugebauer's theory is based on a field theoretical variational principle that calculates the density of the entropy production σ in the general form

$$\sigma = S^i{}_{;i} = -(-g)^{-1/2}(\delta L\sqrt{-g}/\delta V_A)\underset{\xi}{\mathscr{L}}V_A \geqslant 0 \tag{1}$$

(S^i is entropy current density, L the Lagrangian of the irreversible system (see below), $g = \det|g_{ik}|$, V_A is a set of independent covariant state variables (metric g_{ik} included), $\delta/\delta V_A$ is the variational derivative, $\mathscr{L}$ is the Lie derivative with respect to a time-like vector field $\xi^i = u^i/T$ (Tolman's temperature vector), u^i is the four-velocity and T the temperature). In (1) the sets

$$(J_A) = (-(k_A/\sqrt{-g})(\delta L\sqrt{-g}/\delta V_A)) \tag{2}$$

and

$$(X_A) = (k_A{}^{-1} \underset{\xi}{\mathscr{L}} V_A) \tag{3}$$

play the role of generalised thermodynamical currents and forces, respectively (k_A is a constant chosen for convenience). According to Onsager these are connected by linear relations

$$J_A = \sum_B L_{AB} X_B \tag{4}$$

(L_{AB} are kinetic coefficients).

In general not all of the X_A and J_A respectively are independent. Further one should point out that the above separation of σ is not unique but a matter of convenience.

In spirit of Landau and Lifshitz we introduce small fluctuations by adding random terms $\tilde{J}_A$ to the relations (4) which now are replaced by

$$J_A = \sum_B L_{AB} X_B + \tilde{J}_A. \tag{5}$$

In a simplest case of Gaussian processes only the first two correlation functions of $\tilde{J}_A$ are of importance:

$$\langle \tilde{J}_A \rangle = 0, \tag{6}$$

guaranteeing that the mean behaviour of the system is governed by (4), and

$$\langle \tilde{J}_A(x) \tilde{J}_B(\bar{x}) \rangle = 2kcQ_{AB}\delta^4(x, \bar{x}) \tag{7}$$

(k is Boltzmann's constant, c the velocity of light, $x = (x^1, \ldots, x^4)$, $\langle \ldots \rangle$ is the ensemble average, $\delta^4(x, \bar{x})$ the four-dimensional covariant δ-function).

Equation (7) is a result of the well known procedures to obtain general relativistic laws of continuous media by the help of the strong equivalence principle (Misner *et al* 1973). Because the correlations are given by interactions in the atomic and molecular range this principle is applicable in the present case as well as in the whole theory of continuous media.

The δ-function-like behaviour of the second-order correlation functions of the non-relativistic theory both in space and time makes this straightforward generalisation reasonable. The quantities Q_{AB} turn out to be dependent on L_{AB}. Examples will be given below. The general relations (5)–(7) are applicable to fluids and solids, including media with complicated internal structure like superconductors.

3. Fluctuations in hydrodynamics

For one-component isotropic fluids we take as independent variables V_A the metric tensor g_{ik}, the invariant temperature T, and the particle number density ρ (mole/cm^3).

The Lagrangian L in (1) reads (Neugebauer 1977, 1980)

$$L = -R/2\varkappa_0 + \rho f(T, \rho) \tag{8}$$

(R is the curvature invariant, $\varkappa_0$ Einstein's gravitational constant, $f(T, \rho)$ the molar free energy).

Now σ becomes from (1)

$$\sigma = -\tfrac{1}{2}[(R^{ik} - \tfrac{1}{2}g^{ik}R)/\varkappa_0 - \mathring{T}^{ik}]\underset{\xi}{\mathscr{L}} g_{ik} - (\rho u^i)_{;i}\mu/T \geqslant 0. \tag{9}$$

$$\mathring{T}^{ik} = \rho e\, u^i u^k/c^2 + ph^{ik} \tag{10}$$

is the reversible part of the energy–momentum tensor for fluid media (R^{ik} is the Ricci tensor, e the molar internal energy, μ the chemical potential, p the pressure, $h^{ik} = g^{ik} + u^i u^k/c^2$ the spatial metric of a comoving observer).

Because of the conservation of the particle number the term containing $(\rho u^i)_{;i}$ in (9) can be omitted (Neugebauer 1977, 1980). The only independent thermodynamical forces—containing in a covariant manner both internal friction and heat conductivity—are $-\tfrac{1}{2}\mathscr{L}_\xi g_{ik}$.

Now (5) takes the concrete form ($k_A = -2$)

$$(R^{ik} - \tfrac{1}{2}g^{ik}R)/\varkappa_0 - \mathring{T}^{ik} = L^{iklm}(-\tfrac{1}{2}\underset{\xi}{\mathscr{L}} g_{lm}) + \tilde{T}^{ik} \tag{11}$$

with (Neugebauer 1977, 1980)

$$L^{iklm} = \eta T(h^{il}h^{km} + h^{im}h^{kl}) + (\zeta - \tfrac{2}{3}\eta)Th^{ik}h^{lm}$$
$$+ (\varkappa T^2/c^2)(h^{il}u^k u^m + h^{km}u^i u^l + h^{im}u^k u^l + h^{kl}u^i u^m). \tag{12}$$

η, ζ and $\varkappa$ are the coefficients of shear viscocity, bulk viscosity and heat conductivity, respectively.

The RHS of (11) is the irreversible part of the energy–momentum tensor including the random fluctuations $\tilde{T}^{ik}$.

In this framework Einstein's field equations naturally become

$$R^{ik} - \tfrac{1}{2}g^{ik}R = \varkappa_0(T^{ik} + \tilde{T}^{ik}) \equiv \varkappa_0 T^{ik}_{\text{tot}} \tag{13}$$

with the equations of motion

$$T^{ik}_{\text{tot};k} \equiv (T^{ik} + \tilde{T}^{ik})_{;k} = 0. \tag{14}$$

T^{ik} corresponds to the usual energy–momentum tensor for fluids including irreversible processes

$$T^{ik} = \mathring{T}^{ik} - \tfrac{1}{2}L^{iklm}\underset{\xi}{\mathscr{L}} g_{lm}.$$

T^{ik}_{tot} can be decomposed with respect to the four-velocity:

$$T^{ik}_{\text{tot}} = (\rho e/c^2)u^i u^k + c^{-1}q^i u^k/c + c^{-1}q^k u^i/c + t^{ik} \tag{15}$$

(q^k is the heat flow vector, $s^{ik} = -t^{ik}$ the stress tensor,

$$t^{ik} = \pi^{ik} + \pi h^{ik},\; q^i u_i = t^{ik}u_i = \pi^i_i = 0).$$

Analogously $\tilde{T}^{ik}$ splits into

$$\tilde{T}^{ik} = c^{-2}(u^i\tilde{q}^k + u^k\tilde{q}^i) + \tilde{\pi}^{ik} + \tilde{\pi}h^{ik} \tag{16}$$

with

$$\tilde{\pi}^{ik}u_k = \tilde{q}^i u_i = \tilde{\pi}^i{}_i = 0. \tag{17}$$

(A contribution proportional to $u^i u^k$ does not occur because the rest mass conservation and the non-relativistic limiting behaviour exclude this possibility.) From (11), (15), and (16) we get

$$\pi^{ik} = -2\eta\sigma^{ik} + \tilde{\pi}^{ik}, \tag{18}$$

$$\pi - p = -\zeta\Theta + \tilde{\pi}, \tag{19}$$

$$q^k = -\varkappa\theta^k + \tilde{q}^k, \tag{20}$$

with shear

$$\sigma^{ik} = h^{ir}h^{ks}\tfrac{1}{2}(u_{r;s} + u_{s;r} - \tfrac{2}{3}u^n_{;n}h_{rs}), \tag{21}$$

expansion

$$\Theta = u^n{}_{;n} \tag{22}$$

and

$$\theta_k = h^l_k(T_{,l} + T\dot{u}_l/c^2). \tag{23}$$

The correlation functions of $\tilde{\pi}^{ik}$, $\tilde{\pi}$ and $\tilde{q}^k$ can be determined in analogy to the non-relativistic theory. Using the principle of equivalence we obtain (Wulfert 1982):

$$\langle\tilde{\pi}^{ik}\rangle = \langle\tilde{\pi}\rangle = \langle\tilde{q}^k\rangle = 0, \tag{24}$$

$$\langle\tilde{\pi}^{ik}(x)\tilde{\pi}^{lm}(\bar{x})\rangle = 2kcT\eta(h^{il}h^{km} + h^{im}h^{kl} - \tfrac{2}{3}h^{ik}h^{lm})\delta^4(x, \bar{x}), \tag{25}$$

$$\langle\tilde{\pi}(x)\tilde{\pi}(\bar{x})\rangle = 2kcT\zeta\delta^4(x, \bar{x}), \tag{26}$$

$$\langle\tilde{q}^k(x)\tilde{q}^l(\bar{x})\rangle = 2kcT^2\varkappa\delta^4(x, \bar{x})h^{kl}, \tag{27}$$

$$\langle\tilde{\pi}^{ik}\tilde{\pi}\rangle = \langle\tilde{\pi}^{ik}\tilde{q}^l\rangle = \langle\tilde{\pi}\tilde{q}^k\rangle = 0. \tag{28}$$

(For a more detailed justification in the case of gases by the help of relativistic kinetic theory see Zimdahl (1983).)

Equations (13)–(28) are the basic formulae of relativistic hydrodynamic fluctuation theory. One easily verifies that the non-relativistic relations of Landau and Lifshitz follow as a limiting case.

To obtain the correlation functions of the physical quantities $(u^i, \rho, \ldots)$ from (24)–(28) one proceeds as follows (Landau and Lifshitz 1978). The quantities $\tilde{\pi}^{ik}$, $\tilde{q}^k$ and $\tilde{\pi}$ are regarded as known space–time functions. The equations of motion (14) are formally solved. Thus the physical quantities become linear functionals of the $\tilde{\pi}^{ik}$, $\tilde{\pi}$ and $\tilde{q}^k$ and any quadratic or bilinear form of the $(u^i, \rho, \ldots)$ is expressible by quadratic or bilinear functionals of $\tilde{\pi}^{ik}$, $\tilde{\pi}$ and $\tilde{q}^k$. Using (24)–(28) the averaging procedure yields the desired results.

4. Fluctuations in solids

In this section the general theory is applied to a general relativistic solid. The Lagrangian in (1) is given by

$$L = -R/2\varkappa_0 + \rho f(\varepsilon_{ik}, h^{rs}, T). \tag{29}$$

The molar free energy f is a function of the deformation tensor ε_{ik} and the temperature T. The spatial metric h^{ik} of a comoving observer enters because f is an invariant. In close analogy to (9) the explicit calculation of (1) provides

$$\sigma = -\tfrac{1}{2}[(R^{ik} - \tfrac{1}{2}g^{ik}R)/\varkappa_0 - \mathring{T}^{ik}]\underset{\xi}{\mathscr{L}} g_{ik} \geq 0. \tag{30}$$

The reversible part $\mathring{T}^{ik}$ of the energy–momentum tensor is now given by

$$\mathring{T}^{ik} = (\rho e/c^2)u^i u^k - s^{ik}_{(\mathrm{el})} \tag{31}$$

with the elastic stress tensor $s^{ik}_{(\mathrm{el})}$

$$s^{ik}_{(\mathrm{el})} = \rho(\partial f/\partial\varepsilon_{ik} - \varepsilon^i{}_r\partial f/\partial\varepsilon_{rk} - \varepsilon^k{}_r\partial f/\partial\varepsilon_{ri}). \tag{32}$$

Following the general line of the theory (5) now results in

$$(R^{ik} - \tfrac{1}{2}g^{ik}R)/\varkappa_0 - \mathring{T}^{ik} = L^{iklm}_{(s)}(-\tfrac{1}{2}\underset{\xi}{\mathscr{L}} g_{lm}) + \tilde{T}^{ik} \tag{33}$$

which formally coincides with (11).

In most cases one can consider the molar internal energy e as a Taylor expansion with respect to ε_{ik}. For small deformations e reduces to a quadratic form. Treating a simple solid with only two elastic and two viscosity coefficients (λ, μ ; η_{I}, η_{II}) we get

$$e = e_0 + \mu\varepsilon_{ik}\varepsilon^{ik} + (\lambda/2)(\varepsilon_r{}^r)^2 + \dots, \tag{34}$$

$$s^{ik}_{(\mathrm{el})} = 2\hat{\mu}\varepsilon^{ik} + \hat{\lambda}h^{ik}\varepsilon^r{}_r + \dots \qquad (\hat{\mu} = \rho\mu,\ \hat{\lambda} = \rho\lambda). \tag{35}$$

With the tensor of the phenomenological coefficients (Wulfert 1982)

$$L^{iklm}_{(s)} = \eta_{\mathrm{I}}T(h^{il}h^{km} + h^{im}h^{kl}) + \eta_{\mathrm{II}}Th^{ik}h^{lm} + (\varkappa T^2/c^4)(h^{il}u^k u^m + h^{km}u^i u^l + h^{im}u^k u^l + h^{kl}u^i u^m), \tag{36}$$

the appropriately split fluctuational tensor $\tilde{T}^{ik}$

$$\tilde{T}^{ik} = c^{-2}(\tilde{q}^i u^k + \tilde{q}^k u^i) - \tilde{s}^{ik}, \tag{37}$$

and (21)–(23) we arrive at

$$s^{ik}_{(\mathrm{visc})} = 2\eta_{\mathrm{I}}(\sigma^{ik} + \tfrac{1}{3}\Theta h^{ik}) + \eta_{\mathrm{II}}\Theta h^{ik} + \tilde{s}^{ik}, \tag{38}$$

$$q^k = -\varkappa h^{kl}(T_{,l} + T\dot{u}_l/c^2) + \tilde{q}^k, \tag{39}$$

with the correlation properties

$$\langle\tilde{s}_{ik}\rangle = \langle\tilde{q}^k\rangle = 0 \tag{40}$$

$$\langle\tilde{s}^{ik}\tilde{q}^l\rangle = 0, \tag{41}$$

$$\langle\tilde{s}^{ik}(x)\tilde{s}^{lm}(\bar{x})\rangle = 2kTc[\eta_{\mathrm{I}}(h^{il}h^{km} + h^{im}h^{kl}) + \eta_{\mathrm{II}}h^{ik}h^{lm}]\delta^4(x, \bar{x}), \tag{42}$$

$$\langle\tilde{q}^i(x)\tilde{q}^k(\bar{x})\rangle = 2kT^2c\varkappa h^{ik}\delta^4(x, \bar{x}), \tag{43}$$

in close analogy to the case of a fluid.

The equations of motion now are

$$T^{ik}_{\mathrm{tot};k} \equiv (T^{ik} + \tilde{T}^{ik})_{;k} = 0 \tag{44}$$

with

$$T^{ik}_{\text{tot}} = \mathring{T}^{ik} + L^{iklm}_{(s)}(-\tfrac{1}{2}\underset{\xi}{\mathscr{L}}\, g_{lm}) + \tilde{T}^{ik}$$
$$= (\rho e/c^2)u^i u^k - s^{ik}_{(\text{el})} - s^{ik}_{(\text{visc})} + (q^i u^k + q^k u^i)/c^2. \tag{45}$$

Equations (33)–(45) are all the equations we need for treating random fluctuations in simple solids.

5. Fluctuations in a long thin bar

As an application of the developed method we calculate the fluctuations of an idealised Weber bar. In a previous paper (Günther and Salié 1978) a differential equation for the oscillations of such a bar in the field of a gravitational wave was derived and Fourier expansion solutions for special initial value problems were presented. Here we extend these calculations to include small random fluctuations.

The material of the bar is assumed to be insulating (sapphire, glass) so that we can neglect heat conductivity. As a result we shall get the correlation functions for the displacement vector in the field of a gravitational wave and in the static earth field. The non-relativistic part of the correlation functions, which to our knowledge also has never been calculated in a corresponding framework, provides the usual noise. The additional terms due to the influence of the gravitation are of the order of magnitude of the gravitational fields and are negligible in most cases.

Therefore the measurability of the general relativistic first-order effects is restricted by the non-relativistic classical fluctuations. This is the result of an exact general relativistic theory.

The gravitational field consists of two parts, a static one, $f^{(\text{st})}_{ik}$ (field of the earth) and a time-dependent one, $f^{(\text{w})}_{ik}$ (wave):

$$g_{ik} = \eta_{ik} + f_{ik}, \tag{46}$$

$$f_{ik} = f^{(\text{st})}_{ik}(x^\nu) + f^{(\text{w})}_{ik}(x^\nu, t) \tag{47}$$

(Greek indices: 1–3).

The deformations and oscillations due to the gravitational fields are small. Other possible oscillations are considered to be small, too. The deformation tensor obeying compatibility relations has the form (Hernandez 1970, Salié 1976)

$$\varepsilon_{ik} = \tfrac{1}{2}(h_{ik} - h^{(0)}_{ik}) \equiv \tfrac{1}{2}(h_{ik} - z^{(\alpha)}_{,i} z^{(\alpha)}_{,k}). \tag{48}$$

The $z^{(\alpha)}(x^\nu, t)$ are three functions given e.g. by the initial values of the problem. In a suggestive picture they may be regarded as flat space coordinates in an idealised stress free state.

For small deformations due to the gravitational fields one can choose coordinates x^α which differ only by a small amount d^α from $z^{(\alpha)}$ (Günther and Salié 1978)

$$x^\alpha = z^{(\alpha)} + d^\alpha. \tag{49}$$

ε_{ik} then takes the form

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}(d_{\alpha,\beta} + d_{\beta,\alpha} + f_{\alpha\beta}), \tag{50}$$

$$\varepsilon_{i4} = 0. \tag{51}$$

In the weak field approximation d_α plays the role of the displacement vector of the classical theory of elasticity.

Linearising (44) with respect to f_{ik} and d_α provides the following differential equations with fluctuations:

$$\begin{aligned}\hat{\mu}\,\Delta d_\alpha + (\hat{\mu}+\hat{\lambda})d_{\nu,\nu,\alpha} + \eta_{\rm I}\Delta d_{\alpha,t} + (\eta_{\rm I}+\eta_{\rm II})d_{\nu,\nu,\alpha,t} - (\rho e/c^2)d_{\alpha,t,t} \\ = -\tfrac{1}{2}(\hat{\mu}+\hat{\lambda})f^r_{r,\alpha} - \tfrac{1}{2}(\rho e+\hat{\lambda})f_{44,\alpha} + c^{-1}(\rho e-\hat{\mu})f_{\alpha 4,t} \\ -\tfrac{1}{2}(\eta_{\rm I}+\eta_{\rm II})f^r_{r,\alpha,t} - \tfrac{1}{2}\eta_{\rm II}f_{44,\alpha,t} - c^{-1}\eta_{\rm I}f_{\alpha 4,t,t} - \tilde{s}_{\alpha\nu,\nu}.\end{aligned} \tag{52}$$

In the following only the time-dependent part of d_α needs to be considered. The correlation relations of $\tilde{s}_{\alpha\beta}$ are determined by (40) and (42), resulting in

$$\langle \tilde{s}_{\alpha\beta}\rangle = 0, \tag{53}$$

$$\begin{aligned}\langle \tilde{s}_{\alpha\nu}(x)\tilde{s}_{\beta\mu}(\bar{x})\rangle \\ = 2kTc\delta^3(x^\nu-\bar{x}^\nu)\delta(x^4-\bar{x}^4) \\ \times\{[\eta_{\rm I}(\delta_{\alpha\beta}\delta_{\mu\nu}+\delta_{\alpha\mu}\delta_{\beta\nu}) + \eta_{\rm II}\delta_{\alpha\nu}\delta_{\beta\mu}](1-\tfrac{1}{2}f^k{}_k) \\ +\eta_{\rm I}(\delta_{\alpha\beta}f_{\mu\nu}+\delta_{\mu\nu}f_{\alpha\beta}+\delta_{\alpha\mu}f_{\beta\nu}+\delta_{\beta\nu}f_{\alpha\mu}) + \eta_{\rm II}(\delta_{\alpha\nu}f_{\beta\mu}+\delta_{\beta\mu}f_{\alpha\nu})\}.\end{aligned} \tag{54}$$

In our approximation the boundary conditions read (Wulfert 1982)

$$(s^{(\rm el)}_{\alpha\beta} + s^{(\rm visc)}_{\alpha\beta})\eta^\beta = 0 \tag{55}$$

with the normal vector n^i obeying

$$u^i n_i = 0 \quad \text{and} \quad n^i n_i = 1. \tag{56}$$

The idealisation of a long thin bar with length l in the x^1 direction simplifies the system of differential equations (52). By integration over the cross section F of the bar one defines mean quantities, e.g.

$$\bar{d}_\alpha(x^1,t) = F^{-1}\iint_F d_\alpha(x^1,x^2,x^3,t)\,\mathrm{d}x^2\,\mathrm{d}x^3. \tag{57}$$

The linearity of (52) makes this integration possible and yields (using (55)) for the case of a *TT*-gauged gravitational wave

$$\begin{aligned}\hat{\mu}(2\hat{\mu}+3\hat{\lambda})d_{1,1,1} + [\eta_{\rm I}(4\hat{\mu}+3\hat{\lambda})+3\eta_{\rm II}\hat{\mu}]d_{1,1,1,t} + \eta_{\rm I}(2\eta_{\rm I}+3\eta_{\rm II})d_{1,1,1,t,t} \\ -(\rho_0 e_0/c^2)(\hat{\mu}+\hat{\lambda})d_{1,t,t} - (\rho_0 e_0/c^2)(\eta_{\rm I}+\eta_{\rm II})d_{1,t,t,t} \\ = -(\hat{\mu}+\hat{\lambda})\tilde{s}_{11,1} - (\eta_{\rm I}+\eta_{\rm II})\tilde{s}_{11,1,t} + \tfrac{1}{2}\hat{\lambda}\tilde{s}_{AA,1} + \tfrac{1}{2}\eta_{\rm II}\tilde{s}_{AA,1,t}\end{aligned} \tag{58}$$

(bars again omitted, $d_1 = d_1(x^1,t)$, $A, B = 2, 3$; ρ_0 = constant density in an initial state, free of deformation).

Now d_α is expanded in a Fourier series with respect to x^1 and in a Fourier integral with respect to t. A similar expansion is made for the random stress fluctuations averaged in analogy to (57). With (53) and (54) one gets algebraic relations between the expansion coefficients (Wulfert 1982).

If the gravitational wave has the form

$$\begin{aligned}f^{(\rm w)}_{11} = -f^{(\rm w)}_{22} = P\cos(\hat{k}x^3-\Omega t) + Q\sin(\hat{k}x^3-\Omega t) \\ f^{(\rm w)}_{12} = 0, \qquad f^{(\rm w)}_{i3} = f^{(\rm w)}_{i4} = 0\end{aligned} \tag{59}$$

(P, Q are constant, $\hat{k}$ is the wavenumber, Ω is the angular frequency), the correlation functions of the displacement vector are (Wulfert 1982) ($x^1 = x$)

$$\langle d_1(x, t)\rangle = 0, \tag{60}$$

$$\begin{aligned}\langle d_1(x, t)d_1(x', t')\rangle &= \sum_{n=0}^{\infty} \frac{kT}{\pi Fl} \int_{-\infty}^{+\infty} \mathrm{d}\omega \frac{k_n^2}{V^2} \cos[k_n(x - x')] \exp - \mathrm{i}\omega(t - t')]. \\ &\times \{[(\Xi_1 + \omega^2\Xi_2) + [(\Xi_1 + \omega^2\Xi_2)\mathscr{H}_1(t') + \mathrm{i}\omega\mathscr{H}_3(t') + \mathscr{H}_2(t')]/(2\hat{k}r) \\ &+ (\Xi_1 + \omega^2\Xi_2)f_{44}^{\mathrm{st}}/2)]\} \\ &\times [|\omega^3 + \mathrm{i}\omega^2(zk_n^2 + d)/V - \omega k_n^2 b/V - \mathrm{i}ak_n^2/V|]^{-1}\end{aligned} \tag{61}$$

with

$$k_n = \pi n/l, \qquad V = (\rho_0 e_0/c^2)(\eta_{\mathrm{I}} + \eta_{\mathrm{II}}), \qquad a = \hat{\mu}(2\hat{\mu} + 3\hat{\lambda}),$$
$$b = \eta_{\mathrm{I}}(4\hat{\mu} + 3\hat{\lambda}) + 3\eta_{\mathrm{II}}\hat{\mu}, \qquad d = (\rho_0 e_0/c^2)(\hat{\mu} + \hat{\lambda}).$$

r is the width of the bar, $r^2 = F$,

$$\begin{aligned} z &= \eta_{\mathrm{I}}(2\eta_{\mathrm{I}} + 3\eta_{\mathrm{II}}), \\ \Xi_1 &= (2\eta_{\mathrm{I}} + \eta_{\mathrm{II}})\hat{\mu}^2 + \eta_{\mathrm{I}}\hat{\lambda}(4\hat{\mu} + 3\hat{\lambda}), \\ \Xi_2 &= (2\eta_{\mathrm{I}} + \eta_{\mathrm{II}})\eta_{\mathrm{I}}^2 + \eta_{\mathrm{I}}\eta_{\mathrm{II}}(4\eta_{\mathrm{I}} + 3\eta_{\mathrm{II}}). \end{aligned} \tag{62}$$

$\mathscr{H}_i(t') = P\alpha_i \cos(\Omega t' - \delta_i) + Q\beta_i \sin(\Omega t' - \chi_i)$. α_i, β_i, δ_i and χ_i are coefficients consisting of Ω, F, $\hat{\lambda}$, $\hat{\mu}$, η_{I} and η_{II} (see Wulfert 1982).

The first term in the braces of the integral in (61) yields the main contribution to the correlation function of $d_1(x, t)$. This is the classical non-relativistic contribution. The second and third terms are generated by the gravitational wave and by the static field, respectively. In general they are completely negligible. Nevertheless the calculations leading to this result are of importance for the following reasons. Firstly, from a completely general relativistic theory it has been shown that the non-relativistic fluctuation formulae usually introduced *ad hoc* into the theory of gravitational experiments are justified indeed in the weak field approximation. These classical fluctuations provide the limit for the detectibility of the first-order general relativistic effects. Secondly a method is presented which allows us to calculate relativistic corrections to classical correlation functions. This may be of importance in superdense matter in the fields of cosmology and astrophysics.

The integration with respect to ω in (61) can be carried out in the complex ω-plane and we get simple algebraic but rather lengthy formulae (Wulfert 1982), which are investigated for two cases. First we consider an experiment of the Weber type in a satellite (no static earth field) and assume small viscosity coefficients (high quality factor). The ratio between the relativistic and the non-relativistic correlation functions is

$$\frac{\langle d_1(x, t)d_1(x', t')\rangle_{\mathrm{rel}}}{\langle d_1(x, t)d_1(x', t')\rangle} = 1 + (P \cos \Omega t' + Q \sin \Omega t')\left(2 + 3\frac{\hat{\lambda}(\eta_{\mathrm{II}}\hat{\mu} - \eta_{\mathrm{I}}\hat{\lambda})}{(2\eta_{\mathrm{I}} + \eta_{\mathrm{II}})\hat{\mu}^2 + \eta_{\mathrm{I}}\hat{\lambda}(4\hat{\mu} + 3\hat{\lambda})}\right). \tag{63}$$

As expected the corrections due to the gravitational wave are of the order of its magnitude and therefore negligible in most of the interesting cases.

As a realistic example we calculate the non-relativistic correlation function for a bar of a Weber-type experiment in a laboratory on the earth. We consider a bar with length $l = 100$ cm, cross section $F = 1\ \mathrm{cm}^2$, mass density $\rho_0 e_0/c^2 = 2.7\ \mathrm{g/cm}^3$, temperature $T = 4.2$ K and elastic and viscous constants $\hat{\mu} = 27.24 \times 10^{10}\ \mathrm{g/cm\,s}^2$, $\hat{\lambda} = 57.88 \times 10^{10}\ \mathrm{g/cm\,s}^2$, $\eta_{\mathrm{I}} = 1.38 \times 10^{-2}$ g/cm s, $\eta_{\mathrm{II}} = 1.38 \times 10^{-2}$ g/cm s. The series in (61) converges. The first coefficients are proportional to $1/n^2$. For $n > 10^7$ there are deviations but these terms are too small to influence the result. In this case we get as a good approximation for the mean square value

$$\langle d_1(x,t)^2\rangle \approx \frac{1}{\pi^2}\frac{kTl}{F\hat{E}}\sum\frac{1}{n^2} = \frac{1}{6}\frac{kTl}{F\hat{E}} = 1.323\,43\ 10^{-26}\ \mathrm{cm}^2,$$

$$\hat{E} = \hat{\mu}(2\hat{\mu} + 3\hat{\lambda})/(\hat{\mu} + \hat{\lambda}). \tag{64}$$

Direct numerical valuation of (61) by the help of a precise computer program yields the same result. By this program one also shows that a change of η_{I} and η_{II} has only a small influence on the result. For $\eta_{\mathrm{I}} = 10^6\,(\eta_{\mathrm{I}})_{\mathrm{old}}$ and $\eta_{\mathrm{II}} = 0.5 \times 10^6\,(\eta_{\mathrm{I}})_{\mathrm{old}}$ one obtains $\langle d(x,t)^2\rangle = 1.323\,55 \times 10^{-26}\ \mathrm{cm}^2$. This is a deviation of about 0.1‰.

6. Harmonic oscillator approach

In the theoretical treatment of gravitational experiments a bar is often idealised as a harmonic oscillator (Misner *et al* 1973) with the Hamiltonian

$$\mathcal{H} = (2m)^{-1}p^2 + \tfrac{1}{2}k_s x^2.$$

In this case the mean square fluctuations can be calculated very simply by the help of the non-relativistic equipartition theorem:

$$\tfrac{1}{2}k_s\overline{x^2} = \tfrac{1}{2}kT. \tag{65}$$

From Hooke's law

$$K/F = \hat{E}(\Delta l)/l \tag{66}$$

(K is force, $\hat{E}$ Young's modulus, F the cross section, Δl the displacement) the coupling constant k_s can be determined:

$$k_s = K/\Delta l = \hat{E}F/l. \tag{67}$$

For the mean square value $\overline{x^2}$ we get

$$\overline{x^2} = kTl/F\hat{E}. \tag{68}$$

Comparing the results (64) and (68), we conclude that the rough estimations (65)–(68) provide the correct order of magnitude.

The calculations leading to (64) may serve as a justification for using the harmonic oscillator approach for bars in gravitational experiments.

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